Direct Lyapunov Design - A Synthesis Procedure for Motor Schema Using a Second-Order Lyapunov Stability Theorem.

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Abstract

In this paper we propose a new procedure for construction of motor schema typically used in behaviour-based robotics. The procedure reverses the standard stability analysis approach by searching for a control function to fit a pre-defined Lyapunov function. In order to improve the applicability of this procedure, a new Second-Order extension to Lyapunov's Second Method is proposed, allowing a stable schema to be defined directly in terms of actuator force demands. We propose a synthesis procedure called Direct Lyapunov Design, which searches for motor schema maps whose set points satisfy the second-order theorem. The procedure has been applied to a simple Subsumption Architecture controller for an inverted pendulum simulation, yielding stable behaviour.

1 Introduction.

Many behaviour-based robotics architectures have adopted motor schema [1] as the basic solution for defining individual tasks or behaviours. A motor schema is a function mapping sensory input directly to motor/actuator output, consisting of a finite array of set points throughout the state space of the required behaviour control function. A typical two-dimensional motor schema is a map of a velocity field in the state space related to the required behaviour. An example based on Arkin's work [1] is shown in Figure 1.

Behaviour-based systems consist of a number of separate behaviour modules (also known as task modules), each of which implements a different motor schema. All the modules are presented with the same sensory input information simultaneously, and the actuator commands from each one are resolved using an arbitration scheme, which is generally either a priority-based scheme [2] or a vector-summation scheme [3] although other types exist. However, as the intent of this paper is to present a method for designing individual motor schema, we do not discuss further the issues of integrating motor schema into wider system architectures.

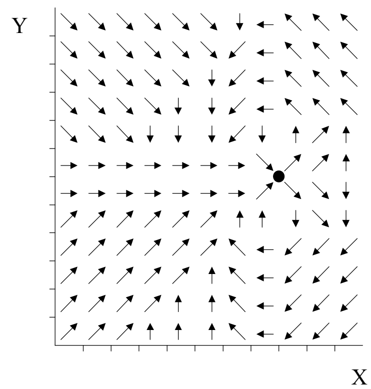

Figure 1: Example Motor Schema for a 'Docking' Behaviour.

A motor schema can be defined as force fields [4] or velocity fields [5]. However, where the schema defines a velocity demand rather than an actuator force, the schema may not generate the intended behaviour in inertia-dominated systems, as their actuators may not be able to overcome system inertia quickly and the velocity demand from the schema cannot instantly be satisfied. While slow-moving wheeled robots can be considered (to a first-order approximation) to have zero inertia, many industrially useful machines (e.g. aircraft, manipulators) cannot be modelled in this way. Therefore, we are interested in applying force-field motor schema.

As a long-term goal, we are interested in designing behaviour-based systems for use in safety-critical applications. Any such applications requires rigorous assurance that the behaviour of systems is safe, and that the system design contains no errors that would lead to unsafe behaviour. Hence, we are interested in proving that the design of any given motor schema generates system behaviour that has goal-seeking properties.

In view of the above research goals, we are therefore investigating the use of Lyapunov stability theory as a mathematical tool for designing or evolving behaviour-based robots. This paper presents some new stability theorems, which extend Lyapunov Stability theory and allow stable motor schema functions to be defined for inertia-dominated systems. Section 2 reviews the standard Lyapunov Stability theory, and Section 3 presents the new extensions (the Second Order Stability Theorems). Section 4 defines a design procedure based on these new theorems, and Section 5 presents some results of an experiment that used the procedure.

2 Lyapunov Stability.

The general definition (from [6]) of marginal and asymptotic stability in the Lyapunov sense is defined below and illustrated in Figure 2.

Marginal stability:

$$\forall R > 0, \quad \exists r > 0, \quad \|\mathbf{x}(0)\| < r \Rightarrow \forall t \geq 0, \quad \|\mathbf{x}(t)\| < R \qquad (1)$$

Asymptotic stability

$$\exists r > 0, \quad \|\mathbf{x}(0)\| < r \Rightarrow \quad t \to \infty, \|\mathbf{x}(t)\| \to 0 \qquad (2)$$

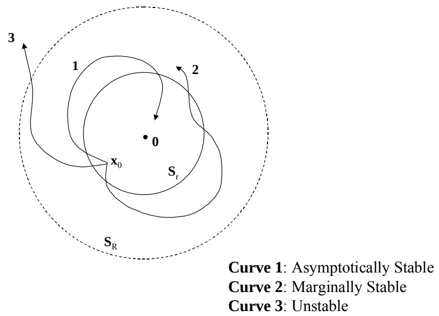

Figure 2: Definitions of Stability.

Of the two forms, asymptotic stability is the most useful since it models actual convergence of a system on its goal state(s). However, Lyapunov's Second Method [7] cannot prove asymptotic stability for every type of trajectory that meets the general specification of Equation (2).

The basic theorem for asymptotic stability is proven only if a positive scalar function (the Lyapunov function) exists, whose derivatives are negative definite along the system state trajectories. This requirement only proves asymptotic stability for trajectories in which the Lyapunov function *V(x)* is always strictly decreasing along any trajectory, as illustrated in Figure 3.

If a system has asymptotically stable trajectories of the type illustrated by Curve 1 in Figure 2, it cannot be proven stable using Lyapunov's Second Method, as the derivative of the Lyapunov Function $\dot{V}(\mathbf{x})$ will be positive at least for part of the trajectory. Hence, the asymptotic stability theorem can only prove the property for a subset of all the systems whose behaviour is stable as defined by equation (2).

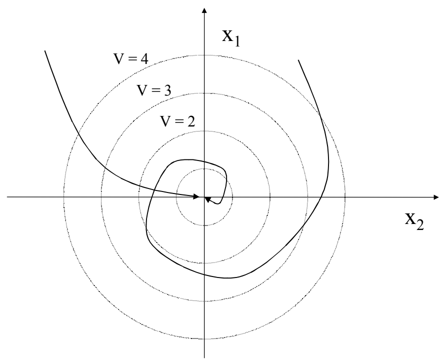

Figure 3: Asymptotic Trajectories Provable with the Standard Theorems.

This issue is a problem for position control tasks in behaviour-based robotics (for example photo-taxis or chemo-taxis) for two reasons. First, physical systems have inertia, which means that the sign (direction) of a system's motion does not change instantaneously. Second, most system actuators generate forces, which govern acceleration rather than velocity, so they cannot govern system velocity in such a way as to make its change sign instantaneously. Therefore, one cannot define a function that ensures negative values of $\dot{V}(\mathbf{x})$, using the standard Lyapunov theorems, if a first-order state vector (containing position states only) is used for problems like photo-taxis or chemo-taxis.

One possible strategy for overcoming this problem might be to define a state vector containing both position and velocity terms. This would ensure that acceleration terms (governable by actuators) would appear in the $\dot{V}(\mathbf{x})$ function, and hence offer a means for a stable behaviour module function to be found. But this is not always possible because, in many cases, the only governable terms in the function will be $\dot{x}_i\ddot{x}_i$ terms. Whenever $\dot{x}_i$ is zero, the contribution of any control action (governing the value of $\ddot{x}_i$) will be cancelled out, and the system will at best only be marginally stable. Where the equilibrium condition is $\boldsymbol{x} = \boldsymbol{0}$, the system will tend to "freeze" because as the $\dot{x}_i$ components converge on zero, they will cancel out the ability of the actuators to change the state of the system. This problem could be overcome with a judicious choice of Lyapunov function that did not have the zero-dynamics properties discussed above. However, selection of a suitable candidate for a Lyapunov function is not straightforward, especially for non-linear systems.

This paper presents some new extensions to the standard Lyapunov stability theorems, called the Second-Order Stability Theorems. The theorems prove the stability of systems whose trajectories may, within limits, diverge temporarily from the equilibrium condition (as illustrated by Curve 1 in Figure 1). These theorems allow the stability of first-order systems to be proven by using a second-order Lyapunov function. This allows problems like taxis-type behaviour patterns to be defined using a state vector consisting purely of position terms. This avoids the zero-dynamics problem by default, and a standard procedure for developing motor schema can be applied.

3 Second-Order Stability Theorems.

To distinguish between the existing stability theorems and the new ones, the former set shall be termed the First-Order Stability Theorems. This is due to the fact that stability is proven in terms of the first derivative $\dot{V}(\mathbf{x})$ of the Lyapunov function. The new theorems are called Second-Order Stability Theorems because they introduce the second derivative $\ddot{V}(\mathbf{x})$ of the Lyapunov function, and allow the stability of a system to be proven within a wider set of constraints than is required for the First-Order Theorems.

3.1 Second-Order Stability Theorems.

The two theorems, for marginal and asymptotic stability, are presented below.

Theorem 1 (Marginal Stability):
Let ***x(t)*** define the time evolution of the system state (i.e. the state trajectory). Let the system be autonomous, i.e. $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$, and let $V(\mathbf{x})$ define a positive definite scalar function over a neighbourhood of $\boldsymbol{x} = \boldsymbol{0}$.

Let $W(t) \equiv V(\mathbf{x}(t))$ denote the value of *V(x) along the system state trajectories*. We assume that $W(t)$ is continuously differentiable twice over any region of interest. Hence $\dot{W}(t) \equiv \dot{V}(\mathbf{x}(t))$ and $\ddot{W}(t) \equiv \ddot{V}(\mathbf{x}(t))$ denote the rate of change and acceleration of the value of *V(x)* along the system's trajectories.

If the state trajectories of a system obey the following formula, the system behaviour is stable:

$$\begin{aligned} &\dot{W}(t) \le 0 \\ &\vee \left\{ \dot{W}(t) > 0 \wedge \dot{W}(t) < \dot{W}_{\max} \wedge \ddot{W}(t) < \ddot{W}_{\max} < 0 \right\} \\ &\Rightarrow \forall R > 0, \exists r > 0, \|\mathbf{x}(0)\| < r \;\rightarrow\; \forall t \ge 0, \|\mathbf{x}(t)\| < R \end{aligned} \tag{3}$$

The concept underlying the second part of the conjunction is that, if the *V(x)* is increasing along the system trajectories, then as long as it is *decelerating* by more than a certain amount, the value of $W(t)$ will remain bounded, and this implies that the equilibrium condition $\boldsymbol{x} = \boldsymbol{0}$ will be stable. This is represented graphically in Figure 4.

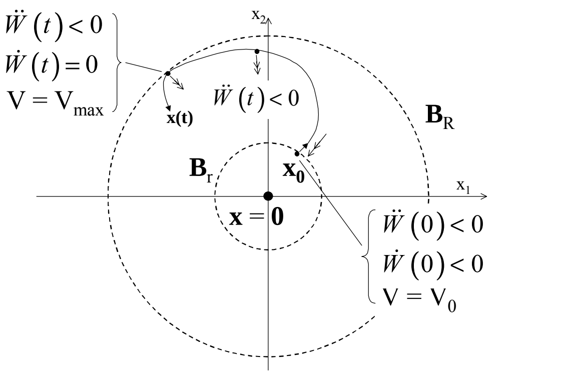

Figure 4: Graphical Representation of Second-Order Stability.

If $\dot{W}(t)$ is allowed to take positive values, but remains bounded by an upper limit $\dot{W}_{\max}$, and $\ddot{W}(t)$ is negative, then the system is stable. The initial $\dot{W}(t)$ term of the above formula is simply a statement of the First-Order Theorem for marginal stability, i.e. that $\dot{W}(t)$ be negative semi-definite (see [6]).

The relationship between V_{max} and V_0 is defined by the following equation:

$$V_{\max} \quad = \quad V_0 + \frac{\dot{W}_{\max}^2}{2\left|\ddot{W}_{\max}\right|} \tag{4}$$

Proof:
A partial proof is provided here, which proves the following property:

$$
\begin{aligned}
&\forall t \geq t_0, \forall V_{\max} > 0, \exists V_0 > 0 : V_0 < V_{\max}, \\
&\exists \dot{W}_{\max} > 0 : \dot{W}(t) \leq \dot{W}_{\max}, \\
&\exists \ddot{W}_{\max} < 0 : \ddot{W}(t) \leq \ddot{W}_{\max} \\
&\Rightarrow \quad W(t_0) \leq V_0 \rightarrow W(t) \leq V_{\max}
\end{aligned} \qquad (5)
$$

Assuming the constraint on $\ddot{W}(t)$ specified in the theorem definition above, the upper limit of the first derivative $\dot{W}(t)$ is defined as:

$$
\begin{aligned}
\sup\left[\dot{W}(t)\right] &= \sup\left[\int_{t_0}^{t} \ddot{W}(t)\,dt + \dot{W}_0\right] \\
&= \int_{t_0}^{t} \sup\left[\ddot{W}(t)\right] dt + \dot{W}_{\max} \qquad (6) \\
&= \dot{W}_{\max} - \left|\ddot{W}_{\max}\right|(t - t_0)
\end{aligned}
$$

Integrating $\dot{W}(t)$ once more yields an equation for $W(t)$. The upper limit of $W(t)$ is therefore

$$\sup\left[W(t)\right] = V_0 + \dot{W}_{\max}(t - t_0) - \tfrac{1}{2}\left|\ddot{W}_{\max}\right|(t - t_0)^2 \quad (7)$$

This upper limit is $V_{\max}$, the highest value of $V(\mathbf{x})$ that any system state trajectory achieves if it satisfies the constraints of (3). The value of this limit can be obtained by investigating the turning points of the function, which occur when the value of equation (6) is zero. Substituting (6) into (7) at the turning point yields equation (4). Since $\ddot{W}(t)$ is always negative by definition, the turning point is a maximum, and therefore defines an upper limit on the value of $W(t)$ and as long as the constraints are satisfied, the system never leaves the neighbourhood defined by (4) ■

A full proof can be achieved by showing that the outer boundary V_{max} proven by equation (5) implies that the state trajectory will remain within ball $\|\mathbf{x}\| \leq R$ as defined in equation (1).

Theorem 2 (Asymptotic Stability):
If the state trajectories of a system obey the following criteria, the system behaviour is asymptotically stable:

$$
\begin{aligned}
&\dot{W}(t) < 0 \\
&\vee \quad \left\{\dot{W}(t) \geq 0 \wedge \dot{W}(t) < \dot{W}_{\max} \wedge \ddot{W}(t) < \ddot{W}_{\max} < 0\right\} \qquad (8) \\
&\Rightarrow \quad \exists r > 0, \quad \|\mathbf{x}(0)\| < r \Rightarrow \quad t \rightarrow \infty, \|\mathbf{x}(t)\| \rightarrow 0
\end{aligned}
$$

Note that, in this theorem, the first $\dot{W}(t)$ term is required to be negative definite, which is the criterion for asymptotic stability for the First-Order theorem.

Proof:
The second term in the conjunction guarantees that for the remaining conditions, a negative value of $\ddot{W}(t)$ is maintained. Therefore, since the value of $\dot{W}(t)$ is modified over time by $\ddot{W}(t)$, it will always eventually become negative ensuring that asymptotic stability is achieved ■

3.2 Notes on the Second-Order Theorems.

The Second-Order Stability Theorems allow the stability of a system to be proven in terms of the second derivative $\ddot{V}(\mathbf{x})$ of the Lyapunov function, which for most systems can be governed directly by a controller with control forces generated by actuators. The theorems can prove the stability of systems that cannot be proven using the First-Order Theorems alone, since trajectories with positive values of $\dot{V}(\mathbf{x})$ are allowed. However, there are some limitations to the theorems, which may serve to limit their application:

1. Stability cannot be proven for arbitrarily large positive values of $\dot{W}(t)$. The control forces provided by the system must be sufficient to overcome any other influence on $\dot{W}(t)$, such as environmental disturbances. Therefore, system behaviour will only be guaranteed or assured if these external influences stay within limits.
2. The dynamics of the system must be defined within the boundary of the upper limit V_{max}. If the constraints on $\dot{W}(t)$ and $\ddot{W}(t)$ are such that V_{max} lies outside the boundaries within which ***f(x)*** has been modelled, then the system trajectories will enter undefined domains for which no valid controller function can be defined. This problem influences the approach taken to the design problem. The following procedure is recommended for use of the Second-Order Theorems:
   - (a) Assert V_{max} to take the value of V at the outer boundary of the domain in which the dynamics is modelled.
   - (b) Apply the theorems to determine whether the controller function meets the criteria of the Second-Order Theorems.

Determine the value of $\dot{W}_{max}$ for the trajectories generated by the controller.

(c) Determine the region of proven stability within the domain modelled by the dynamics ***f(x)***, within which all trajectories are proven stable.

The Direct Lyapunov Design procedure defined in Section 3 follows these general guidelines.

4 Direct Lyapunov Design Procedure.

We have developed a design procedure, called *Direct Lyapunov Design*, for synthesis of motor schema using the Second-Order Stability Theorems. Typically, design procedures using Lyapunov stability theory adopt the approach of defining a control function for a task, and then searching for a Lyapunov function that proves its stability. However, as discussed in [6], one can also adopt the reverse strategy of choosing a candidate scalar function and searching for a controller function that makes it a Lyapunov function by satisfying the stability theorems. We have selected this latter strategy, for the following reasons:

1. Since we are interested in implementing motor schema as piecewise functions, we are not specifying controller transfer functions using analytical equations.
2. We have flexibility in adjusting the perception-action pairs, so it is easier to fit the control function to some pre-defined criteria than vice versa.
3. We must be aware of the limitations discussed in Section 2.2, especially point 2 that addresses the issue of trajectories that leave the modelled domain. It may be difficult to impose these constraints by adopting the conventional design strategy.

The Direct Lyapunov Design procedure is summarised as follows:

(a) Obtain a model of the open-loop dynamics of the system to be controlled. This can take the form of equations, explicitly defined maps, or some other model.

(b) Define the Goal State S required for the task, and the neighbourhood of S for which a task controller is required.

(c) Define a grid or mesh of points over the neighbourhood (each point is a state).

(d) For each point in the grid, select the control action that yields the most stabilising behaviour according to the second-order stability theorems, i.e. the action that yields the most negative value of $\ddot{V}(\mathbf{x})$. Make a record of $\dot{W}_{max}$ and $\ddot{W}_{max}$.

(e) Define a piecewise map function in which the grid points are the central states of each input-output pair, and their associated selected actions are the function outputs.

(f) Define the V_0 region by using equation (4) and substituting the value of $V(\mathbf{x})$ at the outer boundary of the neighbourhood for V_{max}.

It should be noted that strict satisfaction of the second-order theorems would require an infinite number of state/action pairs, one for each point in the neighbourhood of the equilibrium condition, so stability is not strictly proven by this procedure. However, for systems whose trajectories vary reasonably smoothly over the relevant neighbourhoods, the stability properties of the system should be preserved even where piecewise functions are defined with a finite set of input-output pairs.

5 Experimental Results.

The Direct Lyapunov Design procedure has been applied to a simple Subsumption Architecture controlling a simulation of an Inverted Pendulum experiment [8]. The system consisted of three behaviour modules organised into two layers. The system goals were to balance the pole of the inverted pendulum in the upright position, and to move the pendulum cart to the centre of the track. Typical state trajectories, showing pole angle θ and cart position x, are shown in Figure 5.

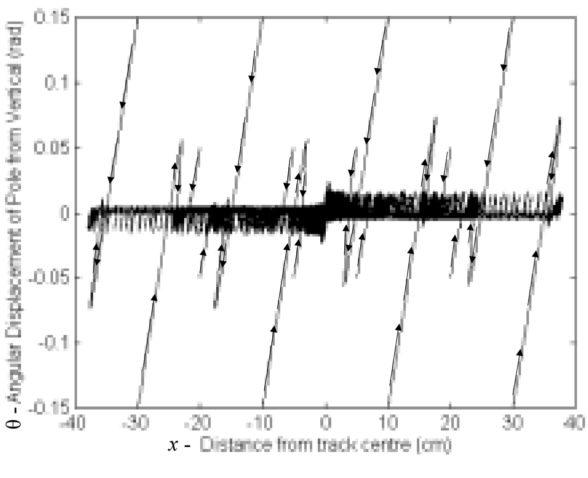

Figure 5: State Trajectories of an Inverted Pendulum Experiment.

The trajectories demonstrate the prioritisation of pole-balancing tasks over cart movement tasks. Their initial motion first converges on a band around θ = 0 (a vertical pole), and subsequently they converge on position $x = 0$ (the centre of the track). These trajectories demonstrate that the motion of the inverted pendulum is asymptotically stable on the state $[\theta, x] = [0,0]$.

6 Discussion.

Further examination of the Second-Order Stability Theorems shows that they provide an explicit mathematical representation of the concept of subsumption in Brooks' Subsumption Architecture [9].

In knowledge-based systems, one production rule 'subsumes' another if the former always evaluates true when the latter does, but is composed of fewer conditional terms. The subsumed rule is usually removed in order to simplify the rule base.

An analogous principle is in effect when using the Second-Order Theorems. Both theorems are of the form where stability is proven either where the natural motion of the system is stable, for example the $\dot{W}(t) < 0$ condition in theorem 2, or if system is forced to be stable by selecting a control function that generates the appropriate values of $\ddot{W}(t)$. Where $\dot{W}(t) < 0$, the natural motion is already stable and *no control actions are required*. Constraints on $\ddot{W}(t)$ are not a necessary criterion for stability, and they can be removed from the control function.

This is a mathematical representation of the principle of parsimonious design as defined by Pfeifer & Scheier [10]. The exploitation of emergent behaviour in the natural (uncontrolled) dynamics of a system, where it is advantageous to do so, is a characteristic feature of many behaviour-based systems.

In addition to the design of individual behaviour modules, we believe that the Second-Order Theorems can be used to form a mathematical basis for designing subsumption architecture. This is the subject of ongoing research.

7 Future Work.

We envisage several directions in which this work could be continued:

1. Higher-order stability theorems could be developed, following the same basic proof approach taken in Section 2. The usefulness of such theorems may be limited, as there would be an increasing number of constraints placed on the structure of problems. However, third-order stability theorems may well be useable in many problems.
2. Since the core of the Direct Lyapunov Design procedure is essentially a search for stabilising control actions, it may be possible to use Genetic Algorithms to improve the efficiency of the search algorithm.
3. It will be important to apply the procedure to more difficult control problems (i.e. of higher dimension), to determine the limits of its applicability and modify it as necessary. We aim to develop behaviour-based controllers for problems such as helicopter flight control or climbing and walking robots.

8 Acknowledgements.

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9 References.

[1] R.C. Arkin, "Motor Schema Based Navigation for a Mobile Robot", *Proc. IEEE Conf. Robotics and Automation*, Raleigh NC, 1987, pp264-271

[2] J.H. Connell, *Minimalist Mobile Robotics: A Colony-style Architecture for an Artificial Creature*, Academic Press, 1990

[3] D. Payton, K. Rosenblatt et al., "Do Whatever Works: A Robust Approach to Fault-Tolerant Autonomous Control", *Jrn. Applied Intelligence 2*, Kluwer Academic Publishers 1992, pp225-250

[4] R.C. Arkin, *Behaviour Based Robotics*, MIT Press 1998, pp98-105

[5] Perry Y. Li, Roberto Horowitz, "Passive Velocity Field Control of Mechanical Manipulators", *IEEE Trans. Robotics and Automation*, Vol.15 No.4 August 1999, pp751-763

[6] Jean-Jacques Slotine, Weiping Li, *Applied Non-linear Control*, Prentice Hall 1991

[7] A.M. Lyapunov, *The General Problem of the Stability of Motion*, Taylor & Francis 1992

[8] Christopher Harper, Alan Winfield, "Designing Behaviour Based Systems Using the Space-time Distance Principle", *Proc. 3rd British Conf. On Autonomous Mobile Robotics and Autonomous Systems (TIMR'01)*, Univ. of Manchester Report UMCS-01-4-1, 2001

[9] Rodney A. Brooks, *Cambrian Intelligence: The Early History of the New AI*, MIT Press, 2000

[10] R. Pfeifer, C. Scheier, "Chapter 10: Design Principles for Autonomous Agents", *Understanding Intelligence*, MIT Press 1999